Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

values involves modification on both sides of the value-situation, both organic interests and extra-organic environment. Education taken in its narrower sense as training the individual student to use his powers and to develop them, emphasizes the former; education taken in a broader sense as including the constant tendency to investigation and experiment and the constant work of remolding great moral and social ideals, emphasizes the latter; and yet the distinction is truly a matter of emphasis.

Here now we have a hint as to the essential nature of the distinction of subjective-objective. If, in the interests of evaluation, of reconstruction of our ideals and ends, we abstract for closer scrutiny those elements of the original value-experience that are most unmistakably connected with the organism's share—that is, with feelings, or desires, or judgments, or dispositions, or presuppositions, or the like—we are giving a statement of the *subjective elements* of the whole value. On the other hand, if we abstract those elements distinctly referable to the extra-organic part—that is, economic commodities, gods, the Altman collection, strawberries in March, posterity, the greatest good of the greatest number, etc.,—we are giving a statement of the *objective elements* of the whole value. Be it remembered that we are not making the *whole real value* either subjective or objective: we are only analyzing it and abstracting different elements for closer scrutiny, and these elements are not and can not be called the value. To have the value as a true value we must have it again in all its unanalyzed dynamic simplicity and immediacy.

It should be remarked that another use of "subjective" is certainly legitimate, namely, its application to the great class of facts found useless, hindering, confusing, and hence not entitled to pews in the congregation of the real. But this use, by emphasizing the personal equation, is really a twin brother of that just given.

In conclusion, then, we may say that a value is neither wholly organic nor extra-organic, but is an experience later analyzable into the two and their interrelation; and that however analyzable, a value still remains a *fact* as unique and primary and important as ever.

J. F. DASHIELL.

PRINCETON UNIVERSITY.

REVIEWS AND ABSTRACTS OF LITERATURE

Principia Mathematica. A. N. WHITEHEAD AND B. RUSSELL. Volume II. Cambridge: University Press. 1912. Pp. xviii + 772.

The second volume of "Principia Mathematica" develops the three subjects of Cardinal Arithmetic, Relation Arithmetic, and Series. The last topic is not concluded. Development of Cardinal Arithmetic is based upon the "Prolegomena" which occupies the last half of Volume I.

Volume II. opens with a prefatory statement of symbolic conventions incident to the "theory of types," after which the authors take up the definition and properties of cardinal numbers. This initial topic is comparatively simple and offers a good example of the manner in which mathematical concepts are derived from simpler logical concepts,—the purpose of "Principia Mathematica."

A cardinal number (say, 5) is to be defined as the class of all classes which are similar to a given class (of five members). To this end similarity has already been defined in terms of one-one relations. Two classes, α and β, are similar when there exists at least one correlating relation which establishes a one-one correspondence of all members of α with all members of β. [$\alpha \,\mathrm{sm}\, \beta \,.\!\equiv\!.\, (\exists R) \,.\, R \in 1 \rightarrow 1 \,.\, \alpha = \text{D‘}R \,.\, \beta = \text{ꓷ‘}R$ (*73 · 1, vol. I.)] All classes similar to a given class of five members will be classes of five, and the class of such classes is the cardinal 5. The cardinal numbers are, then, all those classes of classes which have the relation of similarity.

The oddity of this definition lies in its "extensional" character. The cardinal number of a given class is ordinarily thought of as a *property* of the class, but the attempt so to define cardinal number would rock the "Principia" to its foundations. Throughout the work, the procedure is to determine such properties *in extension*, by logically exhibiting the class of all entities *having* the property. Hence if α be a class of five members, the cardinal number of α is the class of entities having the property "fiveness,"—*i. e.*, the class of all classes similar to α.

0 is the class of classes which are similar to the empty or null class. 1 is the class of classes similar to the class whose only member is x. This last proposition seems to be circular, but the circularity is apparent only and due to "translation" of the symbolism. More accurately, 1 is the class of classes which are similar to the class of those entities which are identical with x,—"identity" having been previously defined without the use of the idea "single" or "only" or "one." The advantages of the symbolism are well exemplified by the fact that the above complicated proposition is simply and accurately expressed as $1 = \text{Nc‘}\iota\text{‘}x$ (*101·2). 2 is the class of all classes which are similar to some class α which has a member x and a remainder (α which is not x) whose cardinal number is 1. [$2 = \hat{\alpha}\{(\exists x) \,.\, x \in \alpha \,.\, \alpha - \iota\text{‘}x \in 1\}$ (*101·301).] Although the authors do not go into the matter, it is obvious that 3, 4, . . ., might be defined by the same method. A class made up of a member, x, and a remainder whose cardinal number was 2 would have the cardinal number 3, and so on.

The further development of cardinal number requires the theory of types. This theory can not be made clear in brief space,—almost one is persuaded it can not be made clear in any space—but something of what it accomplishes may be explained. The usual discussion of the theory of number makes no question of the existence of cardinals in general. But the extensional method of definition here leads to an interesting consequence. Suppose the number of individual things were anything short of infinite,—say 7, for convenience. If the countable things be individuals, then all classes of eight will be empty classes, and the cardinal 8 will not differ from 0. The theory of cardinals beyond 7 would thus be somewhat monotonous. Now the "Principia" does not assume an infinity of individuals; in fact, it does not assume the existence of more than one individual. If the class of individuals has one member, the class of classes of individuals has two members,—the class of one and the empty class. By such a method, the existence of classes of any (given) number of members—and the corresponding cardinals—can be proved for some sufficiently high type of entities,—classes of classes, or classes of classes of classes, or . . . . Thus, by its logical rigor, the theory of types frees mathematics from dependence on empirical data and makes arithmetic intelligible to any rational mind possessed originally of one idea. The theory of types also makes explicit the conditions of significance, or range of meaning, of various propositions, and avoids such contradictions as those discussed in the Introduction to Volume I.

The application of the theory of types leads to "homogeneous cardinals," which are never null, and in terms of these the arithmetical operations can be defined. The idea of a cardinal sum is derived from the logical concept, either . . . or. The logical sum of two classes, α and β, is the class of those things which are members of α or members of β (or members of both),—"either α or β." But in order that the logical sum should represent an arithmetical sum, it is necessary that the classes should be mutually exclusive. To secure this the authors have recourse to a device which is worth illustrating. Suppose we confine our attention to four balls, three of which are red. The cardinal number of the class "red balls" is 3; the cardinal number of "round balls" is 4. The logical sum, "balls that are either red or round," also has the cardinal number 4. But the arithmetical sum of the cardinal number of "red balls" (3) and the cardinal number of "round balls" (4) must be 7, regardless of the fact that there are only four balls in all. To this end we may substitute for "red balls" the class of couples, one member of which is a red ball and the other an *imaginary,* or non-existent, round ball. The number of such couples is the number of red balls. For the class "round balls" substitute the class of couples, one member of which is a round ball and the other an imaginary red ball. If α be the class of couples "red ball with imaginary round ball" and β the class of couples "round ball with imaginary red ball," α and β are always mutually exclusive, whether the red balls are also counted as round balls or not. Thus the *arithmetical* sum of "red balls" and "round balls" is the *logical* sum of α and β; and the concept "$+$" is not a new primitive idea, but is defined in terms of "logical constants." If m be the cardinal number of some class α, and n be the cardinal number of some class β, the cardinal $m + n$ is the class of all classes similar to the arithmetical sum of α and β.

The product $\alpha \times \beta$ is defined as the logical sum (aggregate) of couples which can be formed by taking one member of α with one member of β,—the number of combinations of members of α with members of β. If m be the cardinal number of α and n be the cardinal number of β, the cardinal $m \times n$ is defined as the class of all classes similar to $\alpha \times \beta$.

Exponentiation offers certain logical difficulties. If "m exponent n" were defined in the simplest way as the product of n factors each equal to *m*, propositions about "*m* exponent *n*" would constantly require the "multiplicative axiom,"—that no product is null unless one of its factors is null. This can not be proved for infinite classes. This axiom is not assumed in general, but is inserted as an hypothesis of propositions, proof of which requires it. In order to minimize its use, the authors adopt a different definition of "*m* exponent *n*." To this end, the consideration of a new kind of product, "Prod 'κ," is necessary. "Prod 'κ" represents the possible selections from the class of classes κ. A selected class, or selection, of κ, has been previously defined (*80, Vol. I.) as a class formed by taking one "representative" from each of the constituent classes in κ. Thus if κ is made up of α (with 2 members), β (with 3), and γ (with 4), the possible selections from κ will be 24, the products of its members. Products so defined are not restricted to finite cardinals. In terms of this kind of product, "*m* exponent *n*" is defined as the product of the class of classes-of-couples which can be formed by taking members of *m* with a member of *n*. There are as many such classes-of-couples as there are members of *n*, and each such class has *m* members.

The next section (*113) is devoted to "Greater and Less," after which comes the discussion of "Finite and Infinite." The finite cardinals are treated both as "inductive" and as "non-reflexive." Inductive cardinals are those which can be reached by successive additions of 1. A reflexive class (Cantor's infinite) is similar to a proper part of itself. Inductive and non-reflexive, non-inductive and reflexive, have the same properties in general, only if the multiplicative axiom and the "axiom of infinity"—that all the finite cardinals exist—are both assumed. [The theory of types suffices to prove that any (given) finite cardinal exists, but not that *all* exist].

Inductive cardinals are defined by means of the "ancestral relation," due to Frege (See * 90, Vol. I.). The ancestral relation R^* is related to the relation R as "ancestor of" is related to "parent of." If R is the relation of the cardinal n to cardinal $n+1$, the class to which 0 has the relation R^* will be the inductive cardinals. Through the definition of the ancestral relation, the use of mathematical induction is rendered wholly deductive,—though step-by-step definitions and step-by-step proofs are still necessary in some connections.

To avoid the axiom of infinity, $\aleph_0$, the smallest of the reflexive cardinals (Cantor's transfinite cardinals), is not defined as the cardinal of the class of all the inductive cardinals, but as the class of all classes which can be arranged in a progression.

In concluding the subject of cardinal arithmetic, the authors recapitulate certain conventions and the results of applying the theory of types. In the light of the rigorous discussion which has preceded "we can now adopt the standpoint of ordinary arithmetic, and can for the future in arithmetical operations with cardinals ignore differences of type" (p. 293). The reader's joy in this consummation is clouded only by the fact that we now leave the subject of cardinal number and pass to relation arithmetic.

Relation arithmetic is related to cardinal arithmetic as ordinal similarity is related to similarity in general. Cardinal numbers are classes of similar classes; relation numbers are classes of ordinally similar relations. If ordinal m is the number of a relation, cardinal m is the number of the field (terms) of that relation. The designation "ordinal number" is usually restricted to relations which are serial and well-ordered. Hence relation arithmetic is somewhat more general than ordinal arithmetic and the properties of relation numbers belong to ordinals. The value of this greater generality is something which the reviewer has not yet discovered.

The oddities of extensional definition are nowhere so well displayed as in this part of the work. We hardly think of an ordinal number—"fourth" or "fifth"—as any kind of a relation, though it is an entity determined by its relations. But a relation taken in extension is merely the classes of terms which exhibit that order or relation. Thus "fourth" or "fifth" can be determined in extension only if a fundamental segment which it limits is logically exhibited. Hence a relation number, in "Principia Mathematica," is the number of an aggregate as much as is a cardinal number. A relation number differs from the corresponding cardinal only by requiring the aggregate to have a determined order. Thus the authors depart from the usual conception of ordinal number in ways which they do not remark.

This method of definition also leads to difficulty when we approach the ordinal number one. The "first" (in any ordered set) can not be exhibited in extension unless it have company. The relation "first" must have more than one term. But the ordinal m has cardinal m terms in its field,—ordinal 1, one term. Hence, for the "Principia," there is no ordinal number 1. The nearest approach is the *relation* number of the couple "x with x." But this relation acts like a dyad when added ordinally to another relation. Hence the addition of unity to a relation number is not the addition of the relational 1 and has to be separately treated. Finally the authors *define* the relational sum $1 + 1$ as 2, and frankly state that they do so in order to avoid troublesome exceptions (p. 482). After the logical niceties of the theory of types, this procedure is something of a joke.

On the whole, the treatment of relation arithmetic is a miracle of patience and ingenuity. In spite of greater complexity, the analogy of operations to those of cardinal arithmetic is preserved by clever devices of symbolism. Problems of order, in products and powers, are solved by the "principle of first differences" (due to Hausdorff), the elaboration of which requires much space.

Part V. is devoted to the treatment of serial relations, which are defined as relations of non-identity which are transitive and "connected." A relation P is connected in case, if any two members of its field, x and y, be taken, either xPy or yPx. The development of series contains less that is novel than preceding sections, and discussion of it may well be deferred to the review of Volume III., since that volume is wholly taken up in completing the subject of series.

Throughout the present volume, the authors make use of previous work in the same field. Without such studies as those of Dedekind, Cantor, Frege, and Peano, their development of the subject must have been enormously more difficult, if not impossible. But the "Principia" has a logical rigor not previously attained, and the authors can hardly be said to have "followed" earlier treatises save in the most general way.

Although there is much that is novel in result, still the chief value of the work is its method. Proofs are ordinarily abbreviated or merely indicated, in Volume II., but they can readily be reconstructed, and when so reconstructed are identical in character with those of the "mathematical logic" of Part I. The ordinary "operations of thought" are banished. Mathematical operations are reduced to relations. The only operations used in producing proof are substitution and assertion. Expressions which are equivalent, by definition or proof, are substituted for one another, and values of variables (or expressions which may be regarded as values) are substituted for variables. If by such substitution (in a proposition already assumed or proved) there results an expression of the form "a, b, c implies x, y, z" and "a, b, c" is a previous proposition, "x, y, z" is now asserted as a new lemma or theorem. Adherence to this method is the demonstration of the assertion, first made by Mr. Russell in "Principles of Mathematics," that mathematics may be developed strictly from the fundamental logical relations.

In attempting any thorough study of the volume or of any portion, the reader will do well to familiarize himself with the operation of this method and, by reading the summaries which precede each section, to get the exact meaning of new symbols as they are introduced. He may then turn to more detailed study with profit.

The book is bound to be difficult. Explanatory passages are so condensed that they require more concentrated attention than the portion in symbols. Often one turns from the English to the symbols in order to find out exactly what is meant. Those who are inclined to object to the symbolic method of proof should defer criticism until they are able to read the symbols with understanding.

Whatever one's opinion of logistic or of the particular treatment here given to mathematics, one must at least pay his respects to the logical rigor of the method and the splendid persistence with which it is maintained. The "Principia" is to intellect what the pyramids are to manual labor. And the "Principia" has the added wonder that the whole structure is balanced on its apex of logical constants.

C. I. LEWIS.

UNIVERSITY OF CALIFORNIA.

JOURNALS AND NEW BOOKS.

REVUE PHILOSOPHIQUE. March, 1914. *Hasard et Déterminisme* (pp. 225–265): M. DARBON. – Chance reigns in the part of physical nature which is deprived of all teleological organization; it affects also those conscious beings who do not know how to coordinate their movements in order to attain an end or are hesitating in the choice of ends to pursue: "in both cases, chance appears as a lack of finality" . . . lack of finality meaning a lacuna in the determinism of phenomena. *Valeurs d'Art*